

MEIXNER POLYNOMIALS AND RANDOM PARTITIONS

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Dedicated to our teacher A. A. Kirillov on the occasion of his 70th birthday

ABSTRACT. The paper deals with a 3-parameter family of probability measures on the set of partitions, called the z -measures. The z -measures first emerged in connection with the problem of harmonic analysis on the infinite symmetric group. They are a special and distinguished case of Okounkov's Schur measures. It is known that any Schur measure determines a determinantal point process on the 1-dimensional lattice. In the particular case of z -measures, the correlation kernel of this process, called the discrete hypergeometric kernel, has especially nice properties. The aim of the paper is to derive the discrete hypergeometric kernel by a new method, based on a relationship between the z -measures and the Meixner orthogonal polynomial ensemble. In another paper (Prob. Theory Rel. Fields 135 (2006), 84–152) we apply the same approach to a dynamical model related to the z -measures.

INTRODUCTION

Main definitions and motivations. Recall that a *partition* is an infinite monotone sequence of nonnegative integers, $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots)$, with finitely many nonzero terms λ_i . There is a natural identification of partitions with Young diagrams; for this reason, we denote the set of all partitions by symbol \mathbb{Y} . Clearly, \mathbb{Y} is a countable set. To each partition $\lambda \in \mathbb{Y}$ we assign a weight depending on three parameters z , z' , and ξ . Under suitable restrictions on the parameters (for instance, if z and z' are complex numbers conjugate to each other and $0 < \xi < 1$) all the weights are nonnegative and their sum equals 1. Then we get a probability measure on the set \mathbb{Y} , which makes it possible to speak about *random* partitions. The measures on \mathbb{Y} obtained in this way are called the z -measures and denoted as $M_{z,z',\xi}$ (see section 1 for precise definitions).

Our interest in the z -measures is mainly motivated by the fact that they play a crucial role in harmonic analysis on the infinite symmetric group, see [KOV1], [KOV2], [BO2], [Ol]. On the other hand, for special values of parameters z, z', ξ the z -measures turn into discrete orthogonal polynomial ensembles which in turn are related to interesting probabilistic models: the directed percolation model [Jo1], the stochastic growth model of [GTW], random standard tableaux of rectangular shape [PR]. The z -measures are studied in many research papers: [BO2], [BO3], [BO4], [BO5], [BO6], [BOS], [Ok2]; see also the expository papers [BO1], [Ol]. Finally, note that the z -measures are a particular case of more general objects, the *Schur measures* introduced by Okounkov in [Ok1] and further investigated by many people.

Although the z -measures are quite interesting by themselves, the main problems concern their limits as parameter ξ approaches the critical value 1 (parameters z, z'

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being fixed). Note that, as $\xi \rightarrow 1$, the weight of each partition tends to 0, that is, the measure runs away to infinity. Thus, to catch possible limits we have to embed \mathbb{Y} in a larger space. It turns out that there are different limit regimes, and for each regime the limit measure lives on a suitable space of infinite point configurations (see our paper [BO5] for more details). In other words, the limit measure determines a *random point process*. An appropriate way to describe point processes is to use the language of correlation functions, and the first necessary step is to interpret the initial z -measures as point processes, too.

To this end, we use a well-known interpretation of partitions as *Maya diagrams*, which are semi-infinite point configurations on the 1-dimensional lattice. It is convenient to identify the lattice with the subset $\mathbb{Z}' := \mathbb{Z} + \frac{1}{2} \subset \mathbb{R}$ of (proper) half-integers. Then the Maya diagram of a partition $\lambda \in \mathbb{Y}$ is the configuration (or simply the subset) $\{\lambda_i - i + \frac{1}{2} \mid i = 1, 2, \dots\} \subset \mathbb{Z}'$. Each z -measure $M_{z,z',\xi}$ thus gives rise to a *random* point configuration on \mathbb{Z}' (or a point process on \mathbb{Z}'), and its n th *correlation function* ρ_n ($n = 1, 2, \dots$) expresses the probability $\rho_n(x_1, \dots, x_n)$ that the random configuration contains an arbitrary prescribed finite set of points x_1, \dots, x_n in \mathbb{Z}' .

It is worth noting that the correlation functions survive in various limit regimes, which explains their efficiency.

A remarkable property of the z -measures is that, for any $n = 1, 2, \dots$, the probability $\rho_n(x_1, \dots, x_n)$ can be written as the $n \times n$ determinant $\det[K(x_i, x_j)]$ where $K(x, y)$ is a function on $\mathbb{Z}' \times \mathbb{Z}'$ not depending on n (it depends on parameters z, z', ξ only). Random point processes with such a property are called *determinantal*,¹ and the function $K(x, y)$ is called the *correlation kernel*.

As was first shown in [BO2], the correlation kernel of the z -measure $M_{z,z',\xi}$ can be explicitly written in terms of the Gauss hypergeometric functions; for this reason we called it the *discrete hypergeometric kernel*. Then a number of different proofs were suggested in [Ok1] (see also [Bok]), [Ok2], [BOS]. The goal of the present paper is to better understand the nature of this kernel.

The results. Now we are in a position to describe our main results:

(1) We introduce a system of functions $\psi_a(x) = \psi_a(x; z, z', \xi)$, where the triple (z, z', ξ) is the parameter of the z -measure, x is the argument ranging over \mathbb{Z}' , and $a \in \mathbb{Z}'$ is an additional parameter. For fixed (z, z', ξ) and varying a , the family $\{\psi_a\}$ forms an orthogonal basis in the coordinate Hilbert space $\ell^2(\mathbb{Z}')$. Each function ψ_a can be expressed through the Gauss hypergeometric function.

(2) We exhibit a second order difference operator $D = D(z, z', \xi)$ on \mathbb{Z}' which is diagonalized in the basis $\{\psi_a\}$. The eigenvalue of D corresponding to the eigenfunction ψ_a is equal to $a(1 - \xi)$. (We assume $0 < \xi < 1$, so that the eigenvalue $a(1 - \xi)$ is positive or negative depending on the sign of parameter $a \in \mathbb{Z}'$.)

(3) Set $\mathbb{Z}'_+ = \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots\}$. We prove that the discrete hypergeometric kernel can be written as

$$K(x, y) = \sum_{a \in \mathbb{Z}'_+} \psi_a(x) \psi_a(y),$$

which means that $K(x, y)$ is the kernel (or simply the matrix) of the spectral projection operator in $\ell^2(\mathbb{Z}')$ corresponding to the positive part of the spectrum of

¹This term, introduced in [BO2] and then employed in Soshnikov's expository paper [S], is now widely used. Earlier works used the term "fermion point processes".

D. This formula together with a three-term recurrence relation satisfied by the eigenfunctions ψ_a implies another expression for the kernel:

$$K(x, y) = \frac{\sqrt{zz'\xi}}{1-\xi} \frac{\psi_{-\frac{1}{2}}(x)\psi_{\frac{1}{2}}(y) - \psi_{\frac{1}{2}}(x)\psi_{-\frac{1}{2}}(y)}{x-y}.$$

Thus, $K(x, y)$ is a discrete *integrable* kernel (see [B] for the definition).

(4) The above sum expresses the kernel as a series of products of hypergeometric functions. On the other hand we can represent the kernel by a double contour integral involving elementary functions only.

From these claims one can readily derive all known results concerning the discrete hypergeometric kernel.

The method. Our approach relies on the observation made in [BO2] which relates the z -measures to the Meixner orthogonal polynomials. Assume one of the parameters (z, z') is a positive integer $N = 1, 2, \dots$ while the other parameter is a real number greater than $N - 1$. This is a rather special degenerate case: the weight assigned to a partition λ vanishes unless $\lambda_{N+1} = \lambda_{N+2} = \dots = 0$, so that the relevant partitions λ depend only on the first N coordinates $\lambda_1, \dots, \lambda_N$. It turns out that in this case the random N -point configuration $\{\lambda_1 + N - 1, \lambda_2 + N - 2, \dots, \lambda_N\}$ on the set \mathbb{Z}_+ of nonnegative integers is a well-known object: it is an example of a (discrete) *orthogonal polynomial ensemble*. The orthogonal polynomial ensembles were extensively studied in connection with random matrix theory as well as various discrete probabilistic models, see [De2], [Jo2], [Jo3], [Kö]. In particular, it is well known that they are determinantal processes and their correlation kernels are closely related to the Christoffel–Darboux kernels for the corresponding family of orthogonal polynomials (in our situation these are the classical Meixner polynomials).

The idea of our approach to the z -measures is to regard them as the result of an *analytic continuation* of the Meixner orthogonal polynomial ensembles in parameter N . In particular, our difference operator D on the lattice \mathbb{Z}' comes from the Meixner difference operator on \mathbb{Z}_+ . It is worth noting, however, that the procedure of analytic continuation is rather delicate, because we extrapolate from the discrete values $z = N = 1, 2, \dots$ to continuous values $z \in \mathbb{C}$. It is this analytic continuation procedure that we regard as the main achievement of the present paper. Even though we use it to rederive a known result, in a more complicated dynamical situation (see the next paragraph) this method is crucial for obtaining new results.

Note that instead of the Meixner polynomials one could equally well use the Krawtchouk orthogonal polynomials (see section 4).

Note also that analytic continuation of a correlation kernel off the integral values of a parameter was used in [Ni] in a very different situation. We are grateful to the referee for this remark.

Dynamics. The present paper can be viewed as an introduction to our paper [BO7] where the same approach is applied to studying a dynamical model related to the z -measures. There we derive a dynamical (i.e., time-dependent) version of the discrete hypergeometric kernel, $K(s, x; t, y)$, where x and y are, as before, points of the lattice \mathbb{Z}' while s and t are time variables. We also evaluate the asymptotics of the kernel $K(s, x; t, y)$ in two limit regimes. We refer to [BO7] for more details.

The difference operator D introduced in the present paper plays an important role in the dynamical picture, too. We regard this operator as the key to understanding the nature of the point processes connected to the z -measures.

Plancherel measure. In the limit as z and z' go to ∞ and ξ goes to 0 in such a way that the product $zz'\xi$ converges to a positive number θ , the z -measure $M_{z,z',\xi}$ turns into the poissonized Plancherel measure M_θ with Poisson parameter θ . Our results about the correlation kernel have counterparts for M_θ , see [BOO]. A dynamical model related to M_θ is studied in [BO8].

Organization of the paper. In section 1 we recall the definition of the z -measures and explicitly describe their relationship to the Meixner orthogonal polynomials ensembles. In section 2 we introduce the difference operator D and we study in detail its eigenfunctions $\psi_a(x)$. In section 3 we compute the correlation kernel. In section 4 we briefly discuss the relationship between the z -measures and the Krawtchouk orthogonal polynomial ensembles.

Acknowledgements. Both authors were partially supported by the CRDF grant RUM1-2622-ST-04. The first author (A. B.) was also partially supported by the NSF grant DMS-0402047.

1. Z-MEASURES

As in Macdonald [Ma] we identify partitions and Young diagrams. By \mathbb{Y}_n we denote the set of partitions of a natural number n , or equivalently, the set of Young diagrams with n boxes. By \mathbb{Y} we denote the set of all Young diagrams, that is, the disjoint union of the finite sets \mathbb{Y}_n , where $n = 0, 1, 2, \dots$ (by convention, \mathbb{Y}_0 consists of a single element, the empty diagram \emptyset). Given $\lambda \in \mathbb{Y}$, let $|\lambda|$ denote the number of boxes of λ (so that $\lambda \in \mathbb{Y}_{|\lambda|}$), let $\ell(\lambda)$ be the number of nonzero rows in λ (the length of the partition), and let λ' denote the transposed diagram.

By $\dim \lambda$ we denote the number of standard tableaux of shape λ . A convenient explicit formula for $\dim \lambda$ is

$$\dim \lambda = \frac{|\lambda|!}{\prod_{i=1}^N (\lambda_i + N - i)!} \prod_{1 \leq i < j \leq N} (\lambda_i - i - \lambda_j + j), \quad (1.1)$$

where N is an arbitrary integer $\geq \ell(\lambda)$ (the above expression is stable in N).

We shall need the *generalized Pochhammer symbol* $(z)_\lambda$:

$$(z)_\lambda = \prod_{i=1}^{\ell(\lambda)} (z - i + 1)_{\lambda_i}, \quad z \in \mathbb{C}, \quad \lambda \in \mathbb{Y},$$

where

$$(x)_k = x(x+1) \dots (x+k-1) = \frac{\Gamma(x+k)}{\Gamma(x)}$$

is the conventional Pochhammer symbol. Note that

$$(z)_\lambda = \prod_{(i,j) \in \lambda} (z + j - i)$$

(product over the boxes of λ), which implies at once the symmetry relation

$$(z)_\lambda = (-1)^{|\lambda|} (-z)_{\lambda'}. \quad (1.2)$$

Obviously, if $z = N$, where $N = 1, 2, \dots$, then $(z)_\lambda$ vanishes for all λ with $\ell(\lambda) > N$. Likewise, if $z = -N$ then $(z)_\lambda$ vanishes when $\ell(\lambda') = \lambda_1 > N$.

Definition 1.1. The z -measure with parameters z, z' , and ξ is the (complex) measure $M_{z,z',\xi}$ on the set \mathbb{Y} which assigns to a diagram $\lambda \in \mathbb{Y}$ the weight

$$M_{z,z',\xi}(\lambda) = (1 - \xi)^{zz'} \xi^{|\lambda|} (z)_\lambda (z')_\lambda \left(\frac{\dim \lambda}{|\lambda|!} \right)^2. \quad (1.3)$$

The above expression makes sense for any complex z, z' and any $\xi \in \mathbb{C} \setminus [1, +\infty)$: Indeed, we may assume $-\pi < \arg(1 - \xi) < \pi$ and then we set

$$(1 - \xi)^{zz'} = |1 - \xi|^{zz'} e^{izz' \arg(1 - \xi)}.$$

Note that the weight is invariant under transposition $z \leftrightarrow z'$. Note also the symmetry relation

$$M_{-z, -z', \xi}(\lambda) = M_{z, z', \xi}(\lambda'),$$

which readily follows from (1.2). Finally, note that the z -measures are a particular case of the Schur measures introduced in [Ok1].

Proposition 1.2. *If $0 < \xi < 1$ and parameters z, z' satisfy one of the three conditions listed below, then the z -measure is a probability measure on \mathbb{Y} .*

The conditions are as follows.

- *Principal series:* The numbers z, z' are not real and are conjugate to each other.
- *Complementary series:* Both z, z' are real and are contained in the same open interval of the form $(m, m + 1)$, where $m \in \mathbb{Z}$.
- *Degenerate series:* One of the numbers z, z' (say, z) is a nonzero integer while z' has the same sign and, moreover, $|z'| > |z| - 1$.

Proof. As follows from [BO5, §1], the series $\sum_\lambda M_{z,z',\xi}(\lambda)$ absolutely converges and its sum equals 1 for any complex z, z' and any complex ξ with $|\xi| < 1$. Thus, it suffices to check that the weights are nonnegative under the assumptions listed above. Since $\xi \in (0, 1)$, this means that the product $(z)_\lambda (z')_\lambda$ is nonnegative.

For the principal series, $(z)_\lambda$ and $(z')_\lambda$ are conjugate to each other and do not vanish, and for the complementary series these are both real numbers of the same sign. Thus, their product is always strictly positive.

Examine now the case of the degenerate series. Assume $z = N = 1, 2, \dots$ and $z' > N - 1$. If $\ell(\lambda) \leq N$ then both $(z)_\lambda$ and $(z')_\lambda$ are strictly positive, and if $\ell(\lambda) > N$ then $(z)_\lambda = 0$ so that the weight vanishes. Likewise, if $z = -N$ and $z' < -(N - 1)$ then the weight is strictly positive if $\ell(\lambda') = \lambda_1$ does exceed N , and vanishes otherwise. \square

From now on we assume that the z -measure belongs to one of these three series and is, therefore, a probability measure. Consequently, we may speak about *random* Young diagrams, with reference to the z -measure.

As it is seen from the above proof, for the principal series or the complementary series, the support of the z -measure is the whole set \mathbb{Y} , while for the degenerate series, the support is a proper infinite subset of \mathbb{Y} .

In the remaining part of the section we will describe the relationship between the degenerate series and the Meixner polynomials. We start with a general definition.

Definition 1.2. Let \mathfrak{X} be a discrete subset of \mathbb{R} , finite or countable, and let $W(x)$ be a positive function on \mathfrak{X} . The N -point *orthogonal polynomial ensemble* with weight function W is the random N -point configuration in \mathfrak{X} such that the probability of a particular configuration $x_1 > \dots > x_N$, where $x_1, \dots, x_N \in \mathfrak{X}$, is given by

$$\text{Prob}\{x_1, \dots, x_N\} = \text{const}_N \prod_{i=1}^N W(x_i) \prod_{1 \leq i < j \leq N} (x_i - x_j)^2. \quad (1.4)$$

Here we assume that the cardinality of \mathfrak{X} is no less than N and that

$$\text{const}_N^{-1} := \sum_{x_1 > \dots > x_N} \left\{ \prod_{i=1}^N W(x_i) \prod_{1 \leq i < j \leq N} (x_i - x_j)^2 \right\} < +\infty.$$

For finite \mathfrak{X} , this condition is trivial, and for infinite \mathfrak{X} , it just means that the weight function W has at least $N - 1$ finite first moments.

The term “orthogonal polynomial ensemble” is related to the following well-known fact. Let $P_0 = 1, P_1, P_2, \dots$ be the orthogonal polynomials with weight function W , the result of Gram–Schmidt orthogonalization of $1, x, x^2, \dots$ in the weighted ℓ^2 space $\ell^2(\mathfrak{X}, W)$. Denote by $K_N(x, y)$ the N th Christoffel–Darboux kernel multiplied by $\sqrt{W(x)W(y)}$:

$$K_N(x, y) = \sqrt{W(x)W(y)} \sum_{i=0}^{N-1} \frac{P_i(x)P_i(y)}{\|P_i\|^2},$$

where the norm refers to the weighted ℓ^2 Hilbert space $\ell^2(\mathfrak{X}, W)$. Note that the kernel K_N corresponds to the projection operator in $\ell^2(\mathfrak{X}, W)$ whose range is the linear span of $1, x, \dots, x^{N-1}$. Then we have

Proposition 1.3. *The probability that the random N -point configuration, as specified in Definition 1.2, contains a given n -point set $\{y_1, \dots, y_n\} \subset \mathfrak{X}$ equals the determinant of the $n \times n$ matrix $[K_N(y_i, y_j)]$.*

For a proof, see, e.g., [De2], [Kö, Lemma 2.8]. Note that the determinant automatically vanishes if $n > N$, because the kernel has rank N .

We will be dealing with a concrete example of the weight function. This is the *Meixner weight function*, which is defined on the set $\mathfrak{X} = \mathbb{Z}_+ := \{0, 1, \dots\}$, depends on parameters $\beta > 0$ and $\xi \in (0, 1)$, and is given by

$$W_{\beta, \xi}^{\text{Meixner}}(\tilde{x}) = \frac{(\beta)_{\tilde{x}} \xi^{\tilde{x}}}{\tilde{x}!} = \frac{\Gamma(\beta + \tilde{x}) \xi^{\tilde{x}}}{\Gamma(\beta) \tilde{x}!}, \quad \tilde{x} \in \mathbb{Z}_+$$

(we denote a point of \mathbb{Z}_+ by \tilde{x} instead of x because this notation is used below in §§2–3).

For $N = 1, 2, \dots$, let $\mathbb{Y}(N) \subset \mathbb{Y}$ denote the set of diagrams λ with $\ell(\lambda) \leq N$. The following correspondence is a bijection between diagrams $\lambda \in \mathbb{Y}(N)$ and N -point configurations on \mathbb{Z}_+ :

$$\lambda \longleftrightarrow \{\tilde{x}_1, \dots, \tilde{x}_N\}, \quad \tilde{x}_i = \lambda_i + N - i \quad (i = 1, \dots, N). \quad (1.5)$$

The next fact was pointed out in [BO2]:

Proposition 1.4. *Under correspondence (1.5), the z -measure of the degenerate series with parameters $(z = N, z' = N + \beta - 1, \xi)$, where $\beta > 0$ and $\xi \in (0, 1)$, turns into the N -point Meixner orthogonal polynomial ensemble with parameters (β, ξ) .*

Proof. It suffices to check that if $\ell(\lambda) \leq N$ and $\{\tilde{x}_1, \dots, \tilde{x}_N\}$ is given by (1.5) then the right-hand side of (1.3) can be written as the right-hand side of (1.4) with the Meixner weight function.

By virtue of (1.1),

$$\left(\frac{\dim \lambda}{|\lambda|!} \right)^2 = \frac{\prod_{i < j} (\tilde{x}_i - \tilde{x}_j)^2}{\prod_i (\tilde{x}_i!)^2}.$$

Next, with $z = N$ and $z' = N + \beta - 1$ we have

$$(z)_\lambda = \frac{\prod_{i=1}^N \tilde{x}_i!}{\prod_{i=1}^N (N-i)!}, \quad (z')_\lambda = \frac{\prod_{i=1}^N \Gamma(\beta + \tilde{x}_i)}{\prod_{i=1}^N \Gamma(\beta + N - i)},$$

and

$$\xi^{|\lambda|} = \xi^{-N(N-1)/2} \prod_{i=1}^N \xi^{\tilde{x}_i}.$$

Combining these formulas we get (1.4) with $W = W_{\beta, \xi}^{\text{Meixner}}$ and

$$\text{const}_N = (1 - \xi)^{N(N+\beta-1)} \xi^{-N(N-1)/2} \prod_{i=1}^N \frac{\Gamma(\beta)}{\Gamma(N-i+1)\Gamma(N-i+\beta)}.$$

□

Remark 1.5. Let λ be the random Young diagram distributed according to a z -measure $M_{z, z', \xi}$. Then the number of boxes $|\lambda|$ has the negative binomial distribution on \mathbb{Z}_+ with parameters zz' and ξ :

$$\begin{aligned} \text{Prob}\{|\lambda| = n\} &= \pi_{zz', \xi}(n) \\ &:= (1 - \xi)^{zz'} \cdot \xi^n \cdot \frac{zz'(zz' + 1) \dots (zz' + n - 1)}{n!}, \quad n = 0, 1, 2, \dots, \end{aligned}$$

Conditioned on $|\lambda| = n$, the distribution of λ is a probability measure $M_{z, z'}^{(n)}$ on \mathbb{Y}_n which does not depend on ξ :

$$M_{z, z'}^{(n)}(\lambda) = \frac{(z)_\lambda (z')_\lambda}{zz'(zz' + 1) \dots (zz' + n - 1)} \cdot \frac{(\dim \lambda)^2}{n!}, \quad \lambda \in \mathbb{Y}_n$$

(recall that \mathbb{Y}_n is the set of diagrams with n boxes). This means that the z -measure $M_{z, z', \xi}$ is the mixture of the probability measures $M_{z, z'}^{(n)}$ with varying index $n \in \mathbb{Z}_+$ by means of the negative binomial distribution $\pi_{zz', \xi}$, see [BO2], [BO4]. For applications to harmonic analysis on the infinite symmetric group one needs the measures $M_{z, z'}^{(n)}$ and their scaling limits as $n \rightarrow \infty$, but it turns out that the “mixed” measures $M_{z, z', \xi}$ have much better properties, and the large n limit can be replaced, to a certain extent, by the $\xi \rightarrow 1$ limit transition. This was the starting point of our paper [BO2]. In the present paper we are dealing with the “mixed” measures only.

2. A BASIS IN THE ℓ^2 SPACE ON THE LATTICE AND THE MEIXNER POLYNOMIALS

In this section we examine a nice orthonormal basis in the ℓ^2 space on the 1-dimensional lattice. The elements of this basis are eigenfunctions of a second order difference operator. They can be obtained from the classical Meixner polynomials via analytic continuation with respect to parameters.

Throughout the section we will assume (unless otherwise stated) that parameters z, z' are in the principal series or in the complementary series but not in the degenerate series. In particular, z, z' are not integers.

Consider the lattice of (proper) half-integers

$$\mathbb{Z}' = \mathbb{Z} + \frac{1}{2} = \{\dots, -\frac{5}{2}, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots\}.$$

Elements of \mathbb{Z}' will be denoted by letters x, y .

We introduce a family of functions on \mathbb{Z}' depending on a parameter $a \in \mathbb{Z}'$ and also on our basic parameters z, z', ξ :

$$\begin{aligned} \psi_a(x; z, z', \xi) &= \left(\frac{\Gamma(x + z + \frac{1}{2})\Gamma(x + z' + \frac{1}{2})}{\Gamma(z - a + \frac{1}{2})\Gamma(z' - a + \frac{1}{2})} \right)^{\frac{1}{2}} \xi^{\frac{1}{2}(x+a)} (1 - \xi)^{\frac{1}{2}(z+z')-a} \\ &\times \frac{F(-z + a + \frac{1}{2}, -z' + a + \frac{1}{2}; x + a + 1; \frac{\xi}{\xi-1})}{\Gamma(x + a + 1)}, \quad x \in \mathbb{Z}', \end{aligned} \quad (2.1)$$

where $F(A, B; C; w)$ is the Gauss hypergeometric function.

Let us explain why this expression makes sense. Since, by convention, parameters z, z' do not take integral values, $\Gamma(x + z + \frac{1}{2})$ and $\Gamma(x + z' + \frac{1}{2})$ have no singularities for $x \in \mathbb{Z}'$. Moreover, the assumptions on (z, z') imply that

$$\Gamma(x + z + \frac{1}{2})\Gamma(x + z' + \frac{1}{2}) > 0, \quad \Gamma(z - a + \frac{1}{2})\Gamma(z' - a + \frac{1}{2}) > 0,$$

so that we can take the positive value of the square root in (2.1). Next, since $\xi \in (0, 1)$, we have $\xi/(\xi - 1) < 0$, and as is well known, the function $w \rightarrow F(A, B; C; w)$ is well defined on the negative semi-axis $w < 0$. Finally, although $F(A, B; C; w)$ is not defined at $C = 0, -1, -2, \dots$, the ratio $F(A, B; C; w)/\Gamma(C)$ is well defined for all $C \in \mathbb{C}$.

Note also that the functions $\psi_a(x; z, z', \xi)$ are real-valued. Their origin will be explained below.

Further, we introduce a second order difference operator $D(z, z', \xi)$ on the lattice \mathbb{Z}' , depending on parameters z, z', ξ and acting on functions $f(x)$ (where x ranges over \mathbb{Z}') as follows

$$\begin{aligned} D(z, z', \xi)f(x) &= \sqrt{\xi(z + x + \frac{1}{2})(z' + x + \frac{1}{2})} f(x + 1) \\ &+ \sqrt{\xi(z + x - \frac{1}{2})(z' + x - \frac{1}{2})} f(x - 1) - (x + \xi(z + z' + x)) f(x). \end{aligned}$$

Note that $D(z, z', \xi)$ is a symmetric operator in $\ell^2(\mathbb{Z}')$.

Proposition 2.1. *The functions $\psi_a(x; z, z', \xi)$, where a ranges over \mathbb{Z}' , are eigenfunctions of the operator $D(z, z', \xi)$,*

$$D(z, z', \xi)\psi_a(x; z, z', \xi) = a(1 - \xi)\psi_a(x; z, z', \xi). \quad (2.2)$$

Proof. This equation can be verified using the relation

$$\begin{aligned} w(C - A)(C - B)F(A, B; C + 1; w) - (1 - w)C(C - 1)F(A, B; C - 1; w) \\ + C[C - 1 - (2C - A - B - 1)w]F(A, B; C; w) = 0 \end{aligned}$$

for the Gauss hypergeometric function, see, e.g., [Er, 2.8 (45)]. \square

The next lemma provides us a convenient integral representation for the functions ψ_a .

Lemma 2.2. *For any $A, B \in \mathbb{C}$, $M \in \mathbb{Z}$, and $\xi \in (0, 1)$ we have*

$$\begin{aligned} \frac{F(A, B; M + 1; \frac{\xi}{\xi - 1})}{\Gamma(M + 1)} &= \frac{\Gamma(-A + 1)\xi^{-M/2}(1 - \xi)^B}{\Gamma(-A + M + 1)} \\ &\times \frac{1}{2\pi i} \int_{\{\omega\}} (1 - \sqrt{\xi}\omega)^{A-1} \left(1 - \frac{\sqrt{\xi}}{\omega}\right)^{-B} \omega^{-M} \frac{d\omega}{\omega}. \end{aligned} \quad (2.3)$$

Here $\xi \in (0, 1)$ and $\{\omega\}$ is an arbitrary simple contour which goes around the points 0 and $\sqrt{\xi}$ in the positive direction leaving $1/\sqrt{\xi}$ outside.

Comments. 1. The branch of the function $(1 - \sqrt{\xi}\omega)^{A-1}$ is specified by the convention that the argument of $1 - \sqrt{\xi}\omega$ equals 0 for real negative values of ω , and the same convention is used for the function $\left(1 - \frac{\sqrt{\xi}}{\omega}\right)^{-B}$.

2. Like the Euler integral formula, formula (2.3) does not make evident the symmetry $A \leftrightarrow B$.

3. The right-hand side of formula (2.3) makes sense for $A = 1, 2, \dots$, when $\Gamma(-A + 1)$ has a singularity. Then the whole expression can be understood, e.g., as the limit value as A approaches one of the points $1, 2, \dots$.

Proof. Since both sides of (2.3) are real-analytic functions of ξ we may assume that ξ is small enough. Then we may apply the binomial formula which gives

$$\begin{aligned} &\xi^{-M/2}(1 - \sqrt{\xi}\omega)^{A-1} \left(1 - \frac{\sqrt{\xi}}{\omega}\right)^{-B} \omega^{-M} \\ &= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-A + 1)_k (B)_l}{k! l!} \xi^{(k+l-M)/2} \omega^{k-l-M}. \end{aligned}$$

After integration only the terms with $k = l + M$ survive. It follows that the right-hand side of (2.3) is equal to

$$\frac{(1 - \xi)^B}{\Gamma(-A + M + 1)} \sum_{l \geq \max(0, -M)} \frac{\Gamma(-A + M + 1 + l) (B)_l}{\Gamma(l + M + 1) l!} \xi^l.$$

We may replace the inequality $l \geq \max(0, -M)$ simply by $l \geq 0$ because for negative integral values of M (when we have to start summation from $l = -M$), the terms with $l = 0, \dots, -M - 1$ automatically vanish due to the factor $\Gamma(l + M + 1)$ in the denominator. Consequently, our expression is equal to

$$\frac{(1 - \xi)^B F(-A + 1 + M, B; M + 1; \xi)}{\Gamma(M + 1)} = \frac{F(A, B; M + 1; \frac{\xi}{\xi - 1})}{\Gamma(M + 1)},$$

where we used the transformation formula [Er, 2.9 (4)]. \square

Proposition 2.3. *We have the following integral representations*

$$\begin{aligned} \psi_a(x; z, z', \xi) &= \left(\frac{\Gamma(x + z + \frac{1}{2})\Gamma(x + z' + \frac{1}{2})}{\Gamma(z - a + \frac{1}{2})\Gamma(z' - a + \frac{1}{2})} \right)^{\frac{1}{2}} \frac{\Gamma(z' - a + \frac{1}{2})}{\Gamma(z' + x + \frac{1}{2})} (1 - \xi)^{\frac{z' - z + 1}{2}} \\ &\quad \times \frac{1}{2\pi i} \oint_{\{\omega\}} \left(1 - \sqrt{\xi}\omega\right)^{-z' + a - \frac{1}{2}} \left(1 - \frac{\sqrt{\xi}}{\omega}\right)^{z - a - \frac{1}{2}} \omega^{-x - a} \frac{d\omega}{\omega} \end{aligned} \quad (2.4)$$

and

$$\begin{aligned} \psi_a(x; z, z', \xi)\psi_a(y; z, z', \xi) &= \varphi_{z, z'}(x, y) \\ &\quad \times \frac{1 - \xi}{(2\pi i)^2} \oint_{\{\omega_1\}} \oint_{\{\omega_2\}} \left(1 - \sqrt{\xi}\omega_1\right)^{-z' + a - \frac{1}{2}} \left(1 - \frac{\sqrt{\xi}}{\omega_1}\right)^{z - a - \frac{1}{2}} \\ &\quad \times \left(1 - \sqrt{\xi}\omega_2\right)^{-z + a - \frac{1}{2}} \left(1 - \frac{\sqrt{\xi}}{\omega_2}\right)^{z' - a - \frac{1}{2}} \omega_1^{-x - a} \omega_2^{-y - a} \frac{d\omega_1}{\omega_1} \frac{d\omega_2}{\omega_2} \end{aligned} \quad (2.5)$$

where

$$\varphi_{z, z'}(x, y) = \frac{\sqrt{\Gamma(x + z + \frac{1}{2})\Gamma(x + z' + \frac{1}{2})\Gamma(y + z + \frac{1}{2})\Gamma(y + z' + \frac{1}{2})}}{\Gamma(x + z' + \frac{1}{2})\Gamma(y + z + \frac{1}{2})} \quad (2.6)$$

Here each contour is an arbitrary simple loop, oriented in positive direction, surrounding the points 0 and $\sqrt{\xi}$, and leaving $1/\sqrt{\xi}$ outside. We also use the convention about the choice of argument as in Comment 1 to Lemma 2.2.

Proof. Indeed, (2.4) immediately follows from (2.1) and (2.3). To prove (2.5) we multiply out the integral representation (2.4) for the first function and the same representation for the second function, but with z and z' interchanged. The transposition $z \leftrightarrow z'$ in (2.4) is justified by the fact the initial formula (2.1) is symmetric with respect to $z \leftrightarrow z'$. As a result of this trick the gamma prefactors involving a are completely cancelled out, and we obtain (2.5) \square

Proposition 2.4. *The functions $\psi_a = \psi_a(x; z, z', \xi)$, where a ranges over \mathbb{Z}' , form an orthonormal basis in the Hilbert space $\ell^2(\mathbb{Z}')$.*

Proof. From (2.4) it is not difficult to see that the function $\psi_a(x; z, z', \xi)$ has exponential decay as $x \rightarrow \pm\infty$. Indeed, depending on whether x goes to $+\infty$ or $-\infty$ we arrange the contour in such a way that $|\omega| > 1$ or $|\omega| < 1$, respectively.

In particular, $\psi_a(x; z, z', \xi)$ is square integrable. Since ψ_a is an eigenfunction of a symmetric difference operator whose coefficients have linear growth at $\pm\infty$, and since to different indices a correspond different eigenvalues, we conclude that these functions are pairwise orthogonal in $\ell^2(\mathbb{Z}')$.

Let us show that $\|\psi_a\|^2 = 1$. Take (2.5) with $x = y$. Then the whole expression simplifies because (2.6) turns into 1. Next, in the double contour integral, we replace the variable ω_2 by its inverse. We obtain

$$\begin{aligned} (\psi_a(x; z, z', \xi))^2 &= \frac{1-\xi}{(2\pi i)^2} \oint \oint \left(1 - \sqrt{\xi} \omega_1\right)^{-z'+a-\frac{1}{2}} \left(1 - \sqrt{\xi} \omega_1^{-1}\right)^{z-a-\frac{1}{2}} \\ &\quad \times \left(1 - \sqrt{\xi} \omega_2^{-1}\right)^{-z+a-\frac{1}{2}} \left(1 - \sqrt{\xi} \omega_2\right)^{z'-a-\frac{1}{2}} \left(\frac{\omega_1}{\omega_2}\right)^{-x-a} \frac{d\omega_1}{\omega_1} \frac{d\omega_2}{\omega_2} \end{aligned}$$

To evaluate the squared norm we have to sum this expression over $x \in \mathbb{Z}'$. We split the sum into two parts according to the splitting $\mathbb{Z}' = \mathbb{Z}'_- \cup \mathbb{Z}'_+$. We take as the contours concentric circles such that $|\omega_1| < |\omega_2|$ in the sum over \mathbb{Z}'_- , and $|\omega_1| > |\omega_2|$ in the sum over \mathbb{Z}'_+ . This gives us

$$\sum_{x \in \mathbb{Z}'} (\psi_a(x; z, z', \xi))^2 = \oint \oint_{|\omega_1| < |\omega_2|} \frac{F(\omega_1, \omega_2)}{\omega_2 - \omega_1} \frac{d\omega_1}{\omega_1} \frac{d\omega_2}{\omega_2} + \oint \oint_{|\omega_1| > |\omega_2|} \frac{F(\omega_1, \omega_2)}{\omega_1 - \omega_2} \frac{d\omega_1}{\omega_1} \frac{d\omega_2}{\omega_2}$$

with

$$\begin{aligned} F(\omega_1, \omega_2) &= \frac{1-\xi}{(2\pi i)^2} \left(1 - \sqrt{\xi} \omega_1\right)^{-z'+a-\frac{1}{2}} \left(1 - \sqrt{\xi} \omega_1^{-1}\right)^{z-a-\frac{1}{2}} \\ &\quad \times \left(1 - \sqrt{\xi} \omega_2^{-1}\right)^{-z+a-\frac{1}{2}} \left(1 - \sqrt{\xi} \omega_2\right)^{z'-a-\frac{1}{2}} \omega_1^{\frac{1}{2}-a} \omega_2^{\frac{1}{2}+a}. \end{aligned}$$

Recall that both contours go in positive direction.

Let us transform the second double-contour integral: keeping the second contour fixed we move the first contour inside the second contour. Then we obtain a double-contour integral which cancels the first double-contour integral, plus a single-contour integral arising from the residue of the function $\omega_1 \rightarrow (\omega_1 - \omega_2)^{-1}$:

$$\frac{1-\xi}{2\pi i} \oint F(\omega, \omega) \frac{d\omega}{\omega^2} = \frac{1-\xi}{2\pi i} \oint \frac{d\omega}{(1 - \sqrt{\xi} \omega)(\omega - \sqrt{\xi})} = 1.$$

Thus, we have shown that the functions ψ_a form an orthonormal family in $\ell^2(\mathbb{Z}')$, and it remains to prove that this family is complete. For $x \in \mathbb{Z}'$, let δ_x stand for the delta function at x . Since the functions δ_x form an orthonormal basis, it suffices to check that

$$\sum_{a \in \mathbb{Z}'} ((\delta_x, \psi_a)_{\ell^2(\mathbb{Z}')})^2 = \sum_{a \in \mathbb{Z}'} (\psi_a(x; z, z', \xi))^2 = 1, \quad \forall x \in \mathbb{Z}'.$$

But this follows from the previous claim and the symmetry $a \leftrightarrow x$ established in the next proposition. \square

Proposition 2.5. *The following symmetry relation holds*

$$\psi_a(x; z, z', \xi) = \psi_x(a; -z, -z', \xi).$$

Proof. Using the classical formula

$$\Gamma(A + \tfrac{1}{2})\Gamma(A - \tfrac{1}{2}) = \frac{\pi}{\cos(\pi A)}$$

and the fact that both $x + \frac{1}{2}$ and $a + \frac{1}{2}$ are integers we check that

$$\frac{\Gamma(z + x + \tfrac{1}{2})\Gamma(z' + x + \tfrac{1}{2})}{\Gamma(z - a + \tfrac{1}{2})\Gamma(z' - a + \tfrac{1}{2})} = \frac{\Gamma(-z + a + \tfrac{1}{2})\Gamma(-z' + a + \tfrac{1}{2})}{\Gamma(-z - x + \tfrac{1}{2})\Gamma(-z' - x + \tfrac{1}{2})}.$$

Applying this to (2.1) and using another classical formula,

$$F(A, B; C; w) = (1 - w)^{C-A-B} F(C - A, C - B; C; w),$$

see [Er, (2.9.2)], we get the required relation.

Another way to prove the proposition is to make a change of the variable in integral (2.4):

$$\omega \mapsto \omega' = \frac{\omega - \sqrt{\xi}}{\sqrt{\xi}\omega - 1}.$$

This is an involutive transformation such that $0 \leftrightarrow \sqrt{\xi}$ and $\infty \leftrightarrow 1/\sqrt{\xi}$. As is readily verified, it leads to transformation $(a, x, z, z') \rightarrow (x, a, -z, -z')$. \square

Corollary 2.6. *The functions $\psi_a = \psi_a(x; z, z', \xi)$ satisfy the following three-term relation*

$$\begin{aligned} (1 - \xi)x\psi_a &= \sqrt{\xi(z - a + \tfrac{1}{2})(z' - a + \tfrac{1}{2})} \psi_{a-1} \\ &+ \sqrt{\xi(z - a - \tfrac{1}{2})(z' - a - \tfrac{1}{2})} \psi_{a+1} + (-a + \xi(z + z' - a)) \psi_a. \end{aligned}$$

Proof. Under symmetry $x \leftrightarrow a$ (Proposition 2.4), this turns into the formula stated in Proposition 2.1. Of course, a direct verification is also possible. \square

The formulas of Proposition 2.1 and Corollary 2.6 show that the functions $\psi_a(x; z, z', \xi)$ possess the *bispectrality* property in the sense of [Gr].

Proposition 2.7. *One more symmetry relation holds:*

$$\psi_a(x; z, z', \xi) = (-1)^{x+a} \psi_{-a}(-x; -z, -z', \xi), \quad x, a \in \mathbb{Z}'.$$

Proof. This follows from the relation

$$\begin{aligned} \frac{F(A, B; C; w)}{\Gamma(C)} &= w^{1-C} \frac{\Gamma(A - C + 1)\Gamma(B - C + 1)}{\Gamma(A)\Gamma(B)} \\ &\times \frac{F(A - C + 1, B - C + 1; 2 - C; w)}{\Gamma(2 - C)}, \quad C \in \mathbb{Z}, \end{aligned}$$

see [Er, 2.8 (19)]. Another way is to make a change of the variable, $\omega \mapsto 1/\omega$, in integral (2.4). \square

In the remaining part of the section we will explain how the functions ψ_a are related to the Meixner polynomials.

Let $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$. To denote points of \mathbb{Z}_+ we will use now the symbols \tilde{x}, \tilde{y} , because the letters x, y were already employed to denote points of \mathbb{Z}' . Recall that the Meixner polynomials are the orthogonal polynomials with respect to the weight function

$$W_{\beta, \xi}^{\text{Meixner}}(\tilde{x}) = \frac{(\beta)_{\tilde{x}} \xi^{\tilde{x}}}{\tilde{x}!} = \frac{\Gamma(\beta + \tilde{x}) \xi^{\tilde{x}}}{\Gamma(\beta) \tilde{x}!}, \quad \tilde{x} \in \mathbb{Z}_+, \quad (2.7)$$

on \mathbb{Z}_+ , where $\beta > 0$ and, as before, $\xi \in (0, 1)$. Our notation for these polynomials is $\mathfrak{M}_n(\tilde{x}; \beta, \xi)$. We use the same normalization of the polynomials as in the handbook [KS] (note that in [KS], our parameter ξ is denoted as c).

Set

$$\widetilde{\mathfrak{M}}_n(\tilde{x}; \beta, \xi) = (-1)^n \frac{\mathfrak{M}_n(\tilde{x}; \beta, \xi)}{\|\mathfrak{M}_n(\cdot; \beta, \xi)\|} \sqrt{W_{\beta, \xi}^{\text{Meixner}}(\tilde{x})}, \quad \tilde{x} \in \mathbb{Z}_+, \quad (2.8)$$

where

$$\|\mathfrak{M}_n(\cdot; \beta, \xi)\|^2 = \sum_{\tilde{x}=0}^{\infty} \mathfrak{M}_n^2(\tilde{x}; \beta, \xi) W_{\beta, \xi}^{\text{Meixner}}(\tilde{x}).$$

The factor $(-1)^n$ is introduced for convenience: it will compensate the same factor in formula (2.10) below.

Proposition 2.8. *Drop the assumption that (z, z') is not in the degenerate series, and assume, just on the contrary, that $z = N$ and $z' = N + \beta - 1$, where $N = 1, 2, \dots$ and $\beta > 0$. Then expression (2.1) for the functions $\psi_a(x; z, z', \xi)$ still makes sense provided that the numbers*

$$\tilde{x} := x + N - \frac{1}{2}, \quad n := N - a - \frac{1}{2} \quad (2.9)$$

are in \mathbb{Z}_+ , and in this notation we have

$$\psi_a(x; z, z', \xi) = \widetilde{\mathfrak{M}}_n(\tilde{x}; \beta, \xi).$$

Proof. We start with the expression of the Meixner polynomials through the Gauss hypergeometric function (see [KS, (1.9.1)]):

$$\mathfrak{M}_n(\tilde{x}; \beta, \xi) = F(-n, -\tilde{x}; \beta; \frac{\xi-1}{\xi}).$$

Applying the transformation

$$F(-n, b; \beta; w) = \frac{\Gamma(1-b)\Gamma(\beta)w^n}{\Gamma(\beta+n)} \frac{F(-n, 1-\beta-n; 1-b-n; w^{-1})}{\Gamma(1-b-n)}, \quad n = 0, 1, \dots, \quad \blacksquare$$

we obtain

$$\begin{aligned} \mathfrak{M}_n(\tilde{x}; \beta, \xi) &= \frac{(-1)^n \Gamma(\tilde{x}+1) \Gamma(\beta)}{\Gamma(\beta+n)} \left(\frac{1-\xi}{\xi} \right)^n \frac{F(-n, -\beta-n+1; \tilde{x}+1-n; \frac{\xi}{\xi-1})}{\Gamma(\tilde{x}+1-n)}. \end{aligned} \quad (2.10)$$

Although the first expression for the polynomials looks simpler than the second one, it turns out that only the second expression is suitable for our purposes. Note that (see [KS, (1.9.2)])

$$\|\mathfrak{M}_n(\cdot; \beta, \xi)\|^{-2} = \frac{\xi^n(1-\xi)^\beta \Gamma(\beta+n)}{\Gamma(\beta)\Gamma(n+1)}.$$

From the last two formulas and the definition of $\widetilde{\mathfrak{M}}_n$ we obtain

$$\begin{aligned} \widetilde{\mathfrak{M}}_n(\tilde{x}; \beta, \xi) &= \sqrt{\frac{\Gamma(\tilde{x}+1)\Gamma(\tilde{x}+\beta)}{\Gamma(n+1)\Gamma(n+\beta)}} \xi^{(\tilde{x}-n)/2} (1-\xi)^{(\beta+2n)/2} \\ &\quad \times \frac{F(-n, -\beta-n+1; \tilde{x}+1-n; \frac{\xi}{\xi-1})}{\Gamma(\tilde{x}+1-n)}. \end{aligned}$$

Comparing this with (2.1) and taking into account (2.9) we get the required equality. \square

Thus, our functions ψ_a can be obtained from the Meixner polynomials by the following procedure:

- We replace the initial polynomials \mathfrak{M}_n by the functions $\widetilde{\mathfrak{M}}_n$. This step is quite clear: as a result we get functions which form an orthonormal basis in the ℓ^2 space on \mathbb{Z}_+ with respect to the weight function 1.

- Next, we make a change of the argument. Namely, we introduce an additional parameter $N = 1, 2, \dots$ and we set $x = \tilde{x} - N + \frac{1}{2}$. Then we get orthogonal functions on the subset

$$\{-N + \frac{1}{2}, -N + \frac{3}{2}, -N + \frac{5}{2}, \dots\} \subset \mathbb{Z}',$$

which exhausts the whole \mathbb{Z}' in the limit as N goes to infinity.

- Then we also need a change of the index. Namely, instead of n we have to take $a = N - n - \frac{1}{2}$. We cannot give a conceptual explanation of this transformation, it is dictated by the formulas. Again, the range of the possible values for a becomes larger together with N , and in the limit as $N \rightarrow +\infty$ we get the whole lattice \mathbb{Z}' .

- Finally, we make a (formal) analytic continuation in parameters N and β , using an appropriate analytic expression for the Meixner polynomials (namely, (2.10)).

We hope that this detailed explanation will help the reader to perceive the analytic continuation arguments in section 3.

Of course, instead of the lattice \mathbb{Z}' we could equally well deal with the lattice \mathbb{Z} , and then numerous “ $\frac{1}{2}$ ” would disappear. However, dealing with the lattice \mathbb{Z}' makes main formulas more symmetric.

Remark 2.9. Note that the difference equation of Proposition 2.1 can be obtained via the procedure described above from the classical difference equation satisfied by the Meixner polynomials. This is precisely the way how we have obtained the difference operator D . Likewise, the three-term relation of Corollary 2.6 precisely corresponds to the classical three-term relation for the Meixner polynomials.

3. THE DISCRETE HYPERGEOMETRIC KERNEL

Let \mathfrak{X} be a countable set. By a *point configuration* in \mathfrak{X} we mean any subset $X \subseteq \mathfrak{X}$. Let $\text{Conf}(\mathfrak{X})$ be the set of all point configurations; this is a compact

space. Assume we are given a probability measure on $\text{Conf}(\mathfrak{X})$ so that we can speak about the *random* point configuration in \mathfrak{X} . The *n*th *correlation function* of our probability measure (where $n = 1, 2, \dots$) is defined by

$$\rho_n(x_1, \dots, x_n) = \text{Prob}\{\text{the random configuration contains } x_1, \dots, x_n\},$$

where x_1, \dots, x_n are pairwise distinct points in \mathfrak{X} . The collection of all correlation functions determines the initial probability measure uniquely.

We say that our probability measure is *determinantal* if there exists a function $K(x, y)$ on $\mathfrak{X} \times \mathfrak{X}$ such that

$$\rho_n(x_1, \dots, x_n) = \det [K(x_i, x_j)]_{i,j=1}^n, \quad n = 1, 2, \dots \quad (3.1)$$

It is worth noting that if such a function $K(x, y)$ exists, then it is not unique. Indeed, any “gauge transformation” of the form

$$K(x, y) \rightarrow \frac{f(x)}{f(y)} K(x, y), \quad (3.2)$$

where f is a nonvanishing function on \mathfrak{X} , does not affect the determinants in the right-hand side of (3.1).

Any function $K(x, y)$ satisfying (3.1) will be called a *correlation kernel* of the initial determinantal measure. Two kernels giving the same system of correlation functions will be called *equivalent*.

As in §2, we are dealing with the lattice \mathbb{Z}' of (proper) half-integers. We split it into two parts, $\mathbb{Z}' = \mathbb{Z}'_- \cup \mathbb{Z}'_+$, where \mathbb{Z}'_- consists of all negative half-integers and \mathbb{Z}'_+ consists of all positive half-integers. For an arbitrary $\lambda \in \mathbb{Y}$ we set

$$\underline{X}(\lambda) = \{\lambda_i - i + \frac{1}{2} \mid i = 1, 2, \dots\} \subset \mathbb{Z}'.$$

For instance, $\underline{X}(\emptyset) = \mathbb{Z}'_-$. The set $\underline{X}(\lambda)$ is sometimes called the *Maya diagram* of λ , see, e.g. [MJD].

The correspondence $\lambda \mapsto \underline{X}(\lambda)$ is a bijection between the Young diagrams λ and those (infinite) subsets $X \subset \mathbb{Z}'$ for which the symmetric difference $X \triangle \mathbb{Z}'_-$ is a finite set with equally many points in \mathbb{Z}'_+ and \mathbb{Z}'_- . Note that

$$\underline{X}(\lambda') = -(\mathbb{Z}' \setminus \underline{X}(\lambda)).$$

Using the correspondence $\lambda \mapsto \underline{X}(\lambda)$ we can interpret any probability measure M on \mathbb{Y} as a probability measure on $\text{Conf}(\mathbb{Z}')$. This makes it possible to speak about the correlation functions of M . Our goal is to compute them explicitly for the z -measures.

Now we can state the main results of the paper.

Theorem 3.1. *Under the above correspondence between Young diagrams and Maya diagrams, any z -measure determines a determinantal measure on $\text{Conf}(\mathbb{Z}')$.*

Theorem 3.2. *The correlation kernel of any z -measure $M_{z,z',\xi}$ from the principal or complementary series can be written in the form*

$$\underline{K}_{z,z',\xi}(x,y) = \sum_{a \in \mathbb{Z}'_+} \psi_a(x; z, z', \xi) \psi_a(y; z, z', \xi), \quad x, y \in \mathbb{Z}', \quad (3.3)$$

where the functions ψ_a are defined in (2.1).

Note that the series in the right-hand side is absolutely convergent. Indeed, since $\{\psi_a\}$ is an orthonormal basis in $\ell^2(\mathbb{Z}')$ (Proposition 2.4), this follows from the fact that the series can be written as

$$\sum_{a \in \mathbb{Z}'_+} (\delta_x, \psi_a)(\psi_a, \delta_y),$$

where δ_x stands for the delta-function at point x on the lattice \mathbb{Z}' , and (\cdot, \cdot) denotes the inner product in $\ell^2(\mathbb{Z}')$.

Formula (3.3) simply means that $\underline{K}_{z,z',\xi}(x,y)$ is the matrix of the orthogonal projection operator in $\ell^2(\mathbb{Z}')$ whose range is the subspace spanned by the basis vectors ψ_a with index $a \in \mathbb{Z}'_+ \subset \mathbb{Z}'$.

Theorem 3.3. *The correlation kernel (3.3) can also be written in the form*

$$\underline{K}_{z,z',\xi}(x,y) = \varphi_{z,z'}(x,y) \hat{\underline{K}}_{z,z',\xi}(x,y) \quad (3.4)$$

where, as in (2.6),

$$\varphi_{z,z'}(x,y) = \frac{\sqrt{\Gamma(x+z+\frac{1}{2})\Gamma(x+z'+\frac{1}{2})\Gamma(y+z+\frac{1}{2})\Gamma(y+z'+\frac{1}{2})}}{\Gamma(x+z'+\frac{1}{2})\Gamma(y+z+\frac{1}{2})} \quad (3.5)$$

and

$$\begin{aligned} \hat{\underline{K}}_{z,z',\xi}(x,y) &= \frac{1}{(2\pi i)^2} \oint_{\{\omega_1\}} \oint_{\{\omega_2\}} \frac{(1 - \sqrt{\xi}\omega_1)^{-z'} \left(1 - \frac{\sqrt{\xi}}{\omega_1}\right)^z (1 - \sqrt{\xi}\omega_2)^{-z} \left(1 - \frac{\sqrt{\xi}}{\omega_2}\right)^{z'}}{\omega_1\omega_2 - 1} \\ &\quad \times \omega_1^{-x-\frac{1}{2}} \omega_2^{-y-\frac{1}{2}} d\omega_1 d\omega_2 \end{aligned} \quad (3.6)$$

where $\{\omega_1\}$ and $\{\omega_2\}$ are arbitrary simple contours satisfying the following three conditions:

- both contours go around 0 in positive direction;
- the point $\xi^{1/2}$ is in the interior of each of the contours while the point $\xi^{-1/2}$ lies outside the contours;
- the contour $\{\omega_1^{-1}\}$ is contained in the interior of the contour $\{\omega_2\}$ (equivalently, $\{\omega_2^{-1}\}$ is contained in the interior of $\{\omega_1\}$).

The kernels $\underline{K}_{z,z',\xi}(x,y)$ and $\hat{\underline{K}}_{z,z',\xi}(x,y)$ are equivalent. Namely, they are related by a “gauge transformation”,

$$\hat{\underline{K}}_{z,z',\xi}(x,y) = \frac{f_{z,z'}(x)}{f_{z,z'}(y)} \underline{K}_{z,z',\xi}(x,y), \quad x, y \in \mathbb{Z}',$$

where

$$f_{z,z'}(x) = \frac{\Gamma(x + z' + \frac{1}{2})}{\sqrt{\Gamma(x + z + \frac{1}{2})\Gamma(x + z' + \frac{1}{2})}} \quad (3.7)$$

The kernel $\widehat{\underline{K}}_{z,z',\xi}(x, y)$ can serve as a correlation kernel for the degenerate series as well.

Proof of Theorems 3.1–3.3. We prove these three theorems simultaneously. Let $\rho_n^{(z,z',\xi)}(x_1, \dots, x_n)$ denote the n -point correlation function of $M_{z,z',\xi}$. The proof splits into two parts.

In the first part, we compute $\rho_n^{(z,z',\xi)}$ for special values of the parameters corresponding to the degenerate series: $z = N = 1, 2, \dots$ and $z' = N + \beta - 1$, where $\beta > 0$. Here we use Proposition 1.4. We show that the formula

$$\rho_n^{(z,z',\xi)}(x_1, \dots, x_n) = \det[\underline{K}_{z,z',\xi}(x_i, x_j)]_{i,j=1}^n$$

is valid (in particular, the values of the kernel in the right-hand side are well defined) when $z = N$, $z' = z + \beta - 1$, provided that N is so large that the numbers $x_i + N - \frac{1}{2}$ are nonnegative. Then we check that in that formula, the kernel $\underline{K}_{z,z',\xi}$ can be replaced by the kernel $\widehat{\underline{K}}_{z,z',\xi}$:

$$\rho_n^{(z,z',\xi)}(x_1, \dots, x_n) = \det[\widehat{\underline{K}}_{z,z',\xi}(x_i, x_j)]_{i,j=1}^n$$

In the second part, we extend the latter formula to other admissible values of parameters (z, z') . To do this we show that both sides are analytic functions in parameters (z, z', ξ) . Moreover, these functions are of such a kind that they are uniquely determined by their values at points $(z = N, z' = N + \beta - 1, \xi)$.

We proceed to the detailed proof.

Lemma 3.4. *Let $z = N = 1, 2, \dots$ and $z' = z + \beta - 1$ with $\beta > 0$. Assume that x_1, \dots, x_n lie in the subset $\mathbb{Z}_+ - N + \frac{1}{2} \subset \mathbb{Z}'$, so that the points $\tilde{x}_i := x_i + N - \frac{1}{2}$ are in \mathbb{Z}_+ .*

Then

$$\rho_n^{(z,z',\xi)}(x_1, \dots, x_n) = \det [K_{N,\beta,\xi}^{\text{Meixner}}(\tilde{x}_i, \tilde{x}_j)]_{i,j=1}^n,$$

where

$$K_{N,\beta,\xi}^{\text{Meixner}}(\tilde{x}, \tilde{y}) = \sum_{m=0}^{N-1} \widetilde{\mathfrak{M}}_m(\tilde{x}; \beta, \xi) \widetilde{\mathfrak{M}}_m(\tilde{y}; \beta, \xi), \quad \tilde{x}, \tilde{y} \in \mathbb{Z}_+,$$

and the functions $\widetilde{\mathfrak{M}}_m(\tilde{x}; \beta, \xi)$ are defined in (2.8).

Proof. According to Proposition 1.3, $K_{N,\beta,\xi}^{\text{Meixner}}(\tilde{x}, \tilde{y})$ is the correlation kernel of the N -point Meixner orthogonal polynomial ensemble with parameters β and ξ .

On the other hand, let, as above, $\mathbb{Y}(N)$ denote the set of Young diagrams λ with $\ell(\lambda) \leq N$. Recall the bijective correspondence (1.5)

$$\lambda \mapsto \widetilde{X}(\lambda) = \{\tilde{x}_1, \dots, \tilde{x}_N\} = \{\lambda_1 + N - 1, \lambda_2 + N - 2, \dots, \lambda_N\}$$

between diagrams $\lambda \in \mathbb{Y}(N)$ and N -point configurations in \mathbb{Z}_+ . Comparing the definition of the infinite configuration $\underline{X}(\lambda) \subset \mathbb{Z}'$ with that of the N -point configuration $\widetilde{X}(\lambda)$ we see that

$$\widetilde{X}(\lambda) = (\underline{X}(\lambda) + N - \frac{1}{2}) \cap \mathbb{Z}_+.$$

By Proposition 1.4, under this correspondence, the degenerate z -measure with parameters $z = N$, $z' = N + \beta - 1$, and ξ turns to the N -point Meixner ensemble with parameters β and ξ . This implies our claim. \square

We take (3.3) as the definition of the kernel $\underline{K}_{z,z',\xi}(x, y)$.

Lemma 3.5. *Let $z = N = 1, 2, \dots$ and $z' = z + \beta - 1$ with $\beta > 0$. Assume that x and y lie in the subset $\mathbb{Z}_+ - N + \frac{1}{2} \subset \mathbb{Z}'$, so that $\tilde{x} := x + N - \frac{1}{2}$ and $\tilde{y} := y + N - \frac{1}{2}$ are in \mathbb{Z}_+ .*

Then expression (3.3) for the kernel $\underline{K}_{z,z',\xi}(x, y)$ is well defined and we have

$$\underline{K}_{z,z',\xi}(x, y) = K_{N,\beta,\xi}^{\text{Meixner}}(\tilde{x}, \tilde{y}).$$

Proof. We have to prove that

$$\sum_{a \in \mathbb{Z}'_+} \psi_a(x; z, z', \xi) \psi_a(y; z, z', \xi) = \sum_{m=0}^{N-1} \widetilde{\mathfrak{M}}_m(\tilde{x}; \beta, \xi) \widetilde{\mathfrak{M}}_m(\tilde{y}; \beta, \xi). \quad (3.8)$$

We recall that the functions $\psi_a(x; z, z', \xi)$ were defined under the assumption that both z, z' are not integers. However, as it can be seen from (2.1), each summand in the left-hand side of (3.8) makes sense under the hypotheses of the lemma.

Set

$$a(m) = N - m - \frac{1}{2}, \quad m = 0, 1, \dots, N-1.$$

By Proposition 2.8,

$$\psi_{a(m)}(x; z, z', \xi) = \widetilde{\mathfrak{M}}_m(\tilde{x}; \beta, \xi), \quad \psi_{a(m)}(y; z, z', \xi) = \widetilde{\mathfrak{M}}_m(\tilde{y}; \beta, \xi),$$

which implies that

$$\sum_{a=\frac{1}{2}}^{N-\frac{1}{2}} \psi_a(x; z, z', \xi) \psi_a(y; z, z', \xi) = \sum_{m=0}^{N-1} \widetilde{\mathfrak{M}}_m(\tilde{x}; \beta, \xi) \widetilde{\mathfrak{M}}_m(\tilde{y}; \beta, \xi). \quad (3.9)$$

Finally, observe that

$$\frac{1}{\Gamma(z - a + \frac{1}{2})} \Big|_{a=N+\frac{1}{2}, N+\frac{3}{2}, \dots} = \frac{1}{\Gamma(N - a + \frac{1}{2})} \Big|_{a=N+\frac{1}{2}, N+\frac{3}{2}, \dots} = 0.$$

We conclude that the infinite sum in the left-hand side of (3.8) actually coincides with the finite sum in (3.9). \square

Together with Lemma 3.4 this implies

Corollary 3.6. *Let $z = N = 1, 2, \dots$ and $z' = z + \beta - 1$ with $\beta > 0$. Assume that x_1, \dots, x_n lie in the subset $\mathbb{Z}_+ - N + \frac{1}{2} \subset \mathbb{Z}'$, so that the points $\tilde{x}_i := x_i + N - \frac{1}{2}$ are in \mathbb{Z}_+ .*

Then

$$\rho_n^{(z,z',\xi)}(x_1, \dots, x_n) = \det [\underline{K}_{z,z',\xi}(x_i, x_j)]_{i,j=1}^n.$$

Lemma 3.7. *Assume that*

- *either (z, z') is not in the degenerate series and $x, y \in \mathbb{Z}'$ are arbitrary*
- *or $z = N = 1, 2, \dots$, $z' > N - 1$, and both x, y are in $\mathbb{Z}_+ - N + \frac{1}{2}$.*

Then the kernel $\widehat{K}_{z,z',\xi}(x, y)$ of Theorem 3.3 is related to the kernel $\underline{K}_{z,z',\xi}(x, y)$ by equality (3.4). Equivalently, the kernels are related by the “gauge transformation” (3.2),

$$\widehat{K}_{z,z',\xi}(x, y) = \frac{f_{z,z'}(x)}{f_{z,z'}(y)} \underline{K}_{z,z',\xi}(x, y), \quad (3.10)$$

where $f_{z,z'}$ is defined in (3.7).

Proof. Let us start with expression (3.3) of the kernel $\underline{K}_{z,z',\xi}$ and let us replace each summand by its integral representation (2.5). It is convenient to set $a - \frac{1}{2} = k$ so that as a ranges over \mathbb{Z}'_+ , k ranges over $\{0, 1, 2, \dots\}$. Then we obtain

$$\begin{aligned} \underline{K}_{z,z',\xi}(x, y) &= \varphi_{z,z'}(x, y) \\ &\times \frac{1-\xi}{(2\pi i)^2} \sum_{k=0}^{\infty} \int_{\{\omega_1\}} \int_{\{\omega_2\}} (1 - \sqrt{\xi}\omega_1)^{-z'} \left(1 - \frac{\sqrt{\xi}}{\omega_1}\right)^{z-1} (1 - \sqrt{\xi}\omega_2)^{-z} \left(1 - \frac{\sqrt{\xi}}{\omega_2}\right)^{z'-1} \\ &\times \omega_1^{-x-\frac{1}{2}} \omega_2^{-y-\frac{1}{2}} \left(\frac{(1 - \sqrt{\xi}\omega_1)(1 - \sqrt{\xi}\omega_2)}{(\omega_1 - \sqrt{\xi})(\omega_2 - \sqrt{\xi})} \right)^k \frac{d\omega_1 d\omega_2}{\omega_1 \omega_2}. \end{aligned}$$

We can choose the contours $\{\omega_1\}$ and $\{\omega_2\}$ so that they are contained in the domain $|\omega| > 1$. Since the fractional-linear transformation

$$\omega \mapsto \frac{1 - \sqrt{\xi}\omega}{\omega - \sqrt{\xi}}$$

preserves the unit circle $|\omega| = 1$ and maps its exterior $|\omega| > 1$ into its interior $|\omega| < 1$, we have on the product of the contours a bound of the form

$$\left| \frac{(1 - \sqrt{\xi}\omega_1)(1 - \sqrt{\xi}\omega_2)}{(\omega_1 - \sqrt{\xi})(\omega_2 - \sqrt{\xi})} \right| \leq q < 1.$$

Therefore, we can interchange summation and integration and then sum the arising geometric progression in the integrand:

$$\sum_{k=0}^{\infty} \left(\frac{(1 - \sqrt{\xi}\omega_1)(1 - \sqrt{\xi}\omega_2)}{(\omega_1 - \sqrt{\xi})(\omega_2 - \sqrt{\xi})} \right)^k = \frac{\left(1 - \frac{\sqrt{\xi}}{\omega_1}\right) \left(1 - \frac{\sqrt{\xi}}{\omega_2}\right) \omega_1 \omega_2}{(1 - \xi)(\omega_1 \omega_2 - 1)}.$$

Then we obtain equality (3.4) with integral (3.6), as desired. Finally, we can relax the assumption on the contours: it suffices to assume that $\{\omega_1^{-1}\}$ is strictly contained inside $\{\omega_2\}$, as in the formulation of Theorem 3.3.

It remains to show that (3.4) is equivalent to (3.10). According to (3.5) consider the expression

$$\frac{1}{\varphi_{z,z'}(x, y)} = \frac{\Gamma(x + z' + \frac{1}{2})\Gamma(y + z + \frac{1}{2})}{\sqrt{\Gamma(x + z + \frac{1}{2})\Gamma(x + z' + \frac{1}{2})\Gamma(y + z + \frac{1}{2})\Gamma(y + z' + \frac{1}{2})}}$$

Let us show that

$$\frac{1}{\varphi_{z,z'}(x,y)} = \frac{f_{z,z'}(x)}{f_{z,z'}(y)}$$

Indeed, $1/\varphi_{z,z'}$ has the form

$$\frac{a(x)b(y)}{\sqrt{a(x)b(x)a(y)b(y)}},$$

and our hypotheses imply that $a(x)b(x)$ and $a(y)b(y)$ are real and strictly positive. We also have

$$f_{z,z'}(x) = \frac{a(x)}{\sqrt{a(x)b(x)}}.$$

Therefore, we get

$$\begin{aligned} \frac{f_{z,z'}(x)}{f_{z,z'}(y)} &= \frac{a(x)\sqrt{a(y)b(y)}}{\sqrt{a(x)b(x)}a(y)} = \frac{a(x)a(y)b(y)}{\sqrt{a(x)b(x)a(y)b(y)}a(y)} \\ &= \frac{a(x)b(y)}{\sqrt{a(x)b(x)a(y)b(y)}} = \frac{1}{\varphi_{z,z'}(x,y)}. \end{aligned}$$

□

Corollary 3.8. *Let $z = N = 1, 2, \dots$ and $z' > N - 1$. Then*

$$\rho_n^{(z,z',\xi)}(x_1, \dots, x_n) = \det \left[\widehat{K}_{z,z',\xi}(x_i, x_j) \right]_{i,j=1}^n \quad (3.11)$$

provided that all the points $x_1, \dots, x_n \in \mathbb{Z}'$ lie in the subset $\mathbb{Z}_+ - N + \frac{1}{2} \subset \mathbb{Z}'$.

Proof. Indeed, this follows from Lemma 3.7 and Corollary 3.6. □

This completes the first part of the proof. Now we proceed to the second part.

Lemma 3.9. (i) *Fix an arbitrary set of Young diagrams $\mathcal{D} \subset \mathbb{Y}$. For any fixed admissible pair of parameters (z, z') , the function*

$$\xi \mapsto \sum_{\lambda \in \mathcal{D}} M_{z,z',\xi}(\lambda),$$

which is initially defined on the interval $(0, 1)$, can be extended to a holomorphic function in the unit disk $|\xi| < 1$.

(ii) *Consider the Taylor expansion of this function at $\xi = 0$,*

$$\sum_{\lambda \in \mathcal{D}} M_{z,z',\xi}(\lambda) = \sum_{k=0}^{\infty} G_{k,\mathcal{D}}(z, z') \xi^k.$$

Then the coefficients $G_{k,\mathcal{D}}(z, z')$ are polynomial functions in z, z' . That is, they are restrictions of polynomial functions to the set of admissible values (z, z') .

Proof. (i) Set $\mathcal{D}_n = \mathcal{D} \cap \mathbb{Y}_n$. By the definition of $M_{z,z',\xi}$,

$$\begin{aligned} \sum_{\lambda \in \mathcal{D}} M_{z,z',\xi}(\lambda) &= \sum_{n=0}^{\infty} \left(\sum_{\lambda \in \mathcal{D}_n} M_{z,z'}^{(n)}(\lambda) \right) \pi_{zz',\xi}(n) \\ &= (1 - \xi)^{zz'} \sum_{n=0}^{\infty} \left(\sum_{\lambda \in \mathcal{D}_n} M_{z,z'}^{(n)}(\lambda) \right) \frac{(zz')_n \xi^n}{n!}. \end{aligned}$$

Each interior sum is nonnegative and does not exceed 1. On the other hand,

$$\sum_{n=0}^{\infty} |\pi_{zz',\xi}(n)| = |1 - \xi|^{zz'} \sum_{n=0}^{\infty} \frac{(zz')_n |\xi|^n}{n!} < \infty, \quad \xi \in \mathbb{C}, \quad |\xi| < 1.$$

This proves the first claim.

(ii) By (1.11),

$$\sum_{\lambda \in \mathcal{D}} M_{z,z',\xi}(\lambda) = (1 - \xi)^{zz'} \sum_{n=0}^{\infty} \sum_{\lambda \in \mathcal{D}_n} (z)_\lambda (z')_\lambda \xi^n \left(\frac{\dim \lambda}{n!} \right)^2.$$

It follows that

$$G_{k,\mathcal{D}}(z, z') = \sum_{n=0}^k \frac{(-zz')_{k-n}}{(k-n)!} \sum_{\lambda \in \mathcal{D}_n} (z)_\lambda (z')_\lambda \left(\frac{\dim \lambda}{n!} \right)^2.$$

Since each \mathcal{D}_n is a finite set, this expression is a polynomial in z, z' . \square

Now we can complete the proof of the theorems. Fix n and an arbitrary n -point subset $X = \{x_1, \dots, x_n\} \subset \mathbb{Z}'$, and regard $\rho_n^{(z,z',\xi)}(x_1, \dots, x_n)$ as a function of parameters z, z', ξ . We want to show that equality (3.11) holds for any admissible (z, z') . Apply Lemma 3.11 to the set \mathcal{D} of those diagrams λ for which $\underline{X}(\lambda)$ contains X , and observe that

$$\rho_n^{(z,z',\xi)}(x_1, \dots, x_n) = \sum_{\lambda \in \mathcal{D}} M_{z,z',\xi}(\lambda).$$

It follows that $\rho_n^{(z,z',\xi)}(x_1, \dots, x_n)$ is a real-analytic function of $\xi \in (0, 1)$ which admits a holomorphic extension to the open unit disk $|\xi| < 1$. Moreover, the Taylor coefficients of this function depend on z, z' polynomially.

On the other hand, from the expression (3.6) for the kernel $\widehat{K}_{z,z',\xi}(x, y)$ it follows that this kernel (and hence the right-hand side of (3.11)) has the same property, with ξ replaced by $\sqrt{\xi}$.

Thus, both sides of (3.11) can be viewed as (restrictions of) holomorphic functions in $\sqrt{\xi}$ with polynomial Taylor coefficients. Since the set

$$\{(z, z') \mid z \text{ is a large natural number } N \text{ and } z' > N - 1\}$$

is a set of uniqueness for polynomials in two variables, we conclude that equality (3.11) is true for any admissible (z, z') .

This proves Theorem 3.1 and Theorem 3.3. Now, Theorem 3.2 follows from Theorem 3.3 and Lemma 3.7. \square

Proposition 3.10. *Formula (3.3) for the kernel can also be written as*

$$\underline{K}_{z,z',\xi}(x,y) = \frac{\sqrt{zz'\xi}}{1-\xi} \frac{\psi_{-\frac{1}{2}}(x)\psi_{\frac{1}{2}}(y) - \psi_{\frac{1}{2}}(x)\psi_{-\frac{1}{2}}(y)}{x-y}. \quad (3.12)$$

Comment. The indeterminacy $0/0$ arising on the diagonal $x = y$ is resolved as follows. Observe that the defining analytic expression (2.1) for $\psi_a(x)$ makes sense for any complex x sufficiently close to the lattice \mathbb{Z}' , so that we may view $\psi_{\frac{1}{2}}(\cdot)$ and $\psi_{-\frac{1}{2}}(\cdot)$ as analytic functions in a neighborhood of $\mathbb{Z}' \subset \mathbb{C}$. Since the numerator in (3.12) is an analytic function in (x, y) vanishing on the diagonal $x = y$, it can be divided by $x - y$. Thus, the value of (3.12) on the diagonal can be computed, say, using the analytic expression (2.1) and the l'Hospital rule.

Proof. Assume first $x \neq y$. Then it suffices to prove that

$$(x-y) \sum_{a \in \mathbb{Z}'_+} \psi_a(x)\psi_a(y) = \frac{\sqrt{zz'\xi}}{1-\xi} (\psi_{-\frac{1}{2}}(x)\psi_{\frac{1}{2}}(y) - \psi_{\frac{1}{2}}(x)\psi_{-\frac{1}{2}}(y)).$$

Recall the three-term relation from Corollary 2.6, which we can write as

$$x\psi_a(x) = A(a, a+1)\psi_{a+1}(x) + A(a, a)\psi_a(x) + A(a, a-1)\psi_{a-1}(x)$$

with appropriate coefficients $A(\cdot, \cdot)$. Using this and the similar relation for $y\psi_a(y)$, and taking into account the symmetry relation

$$A(a, a \pm 1) = A(a \pm 1, a)$$

(which follows from the explicit expression in Corollary 2.6), we readily get, after obvious cancellations,

$$(x-y) \sum_{a \in \mathbb{Z}'_+} \psi_a(x)\psi_a(y) = A(\tfrac{1}{2}, -\tfrac{1}{2})(\psi_{-\frac{1}{2}}(x)\psi_{\frac{1}{2}}(y) - \psi_{\frac{1}{2}}(x)\psi_{-\frac{1}{2}}(y)).$$

Since

$$A(\tfrac{1}{2}, -\tfrac{1}{2}) = \frac{\sqrt{zz'\xi}}{1-\xi},$$

we are done. Notice that the infinite sums involved in this computation are convergent, because, for fixed x and y , $\psi_a(x)$ and $\psi_a(y)$ decay exponentially as $a \rightarrow +\infty$: Indeed, by virtue of Proposition 2.5 this fact reduces to that pointed out in the beginning of proof of Proposition 2.4.

To handle the case $x = y$ we use the same trick as in Lemma 3.9 and the subsequent argument: the Taylor expansion at 0 with respect to variable $\eta := \sqrt{\xi}$.

Specifically, let us regard $\psi_a(x) = \psi_a(x; z, z', \xi)$ as a function in $x \in \mathbb{Z}'$, $a \in \mathbb{Z}'_+$, and $\eta = \sqrt{\xi}$. From the integral representation (2.4) it is clear that this function is well defined as an analytic function in η ranging in the open unit disc $|\eta| < 1$ in \mathbb{C} . The same argument as above shows that this function decays exponentially as $a \rightarrow +\infty$, uniformly on compact subsets of the disc. It follows that the series

$$\underline{K}_{z,z',\xi}(x,y) = \sum_{a \in \mathbb{Z}'_+} \psi_a(x; z, z', \xi) \psi_a(y; z, z', \xi) \quad (3.13)$$

is analytic in the same disc $|\eta| < 1$, too.

On the other hand, from (2.1) it follows that if $x + a \geq 0$ then $\psi_a(x; z, z', \xi)$ is of order $O(\eta^{x+a})$ about $\eta = 0$.² Therefore, expanding the kernel in the Taylor series at $\eta = 0$,

$$\underline{K}_{z,z',\xi}(x, y) = \sum_{n=0}^{\infty} F_n(x, y; z, z') \eta^n,$$

we see that only finitely many terms in the series (3.13) contribute to any fixed coefficient $F_n(x, y; z, z')$. Looking again at (2.1) we see that each coefficient can be written as

$$\begin{aligned} F_n(x, y; z, z') \\ = \frac{\sqrt{(\Gamma(z+x+\frac{1}{2})\Gamma(z'+x+\frac{1}{2})\Gamma(z+y+\frac{1}{2})\Gamma(z'+y+\frac{1}{2}))}}{\Gamma(x+\frac{1}{2})\Gamma(y+\frac{1}{2})} G_n(x, y; z, z'), \end{aligned}$$

where $G_n(x, y; z, z')$ is a *rational* function in (x, y) .

It follows that once we know the kernel out of the diagonal $x = y$ we can extend it to the diagonal uniquely, by an obvious extension of the rational functions $G_n(x, y; z, z')$. Finally, viewing the right-hand side of (3.12) as an analytic function in three variables x , y , and η , it is readily checked that the recipe of extension suggested in the comment to the statement of the Proposition is the correct one. \square

Remark 3.11. 1. The correlation functions of the z -measures $M_{z,z',\xi}$ were first computed in [BO2] in a different form: in that paper we dealt with another embedding of partitions into the set of lattice point configurations. The kernel $\underline{K}_{z,z',\xi}(x, y)$ with $x, y > 0$ coincides with one of the “blocks” of the kernel considered in [BO2]. The relation between both kernels is discussed in detail in [BO5]. The proofs in [BO2] and [BO5] are very different from the arguments of the present section.

2. Two other derivations of the kernel $\underline{K}_{z,z',\xi}(x, y)$ are given in Okounkov’s papers [Ok2] and [Ok1]. In both these papers, the correlation functions are expressed through the vacuum state expectations of certain operators in the infinite wedge Fock space. A (substantial) difference between the methods of [Ok2] and [Ok1] consists in the concrete choice of operators. The general formalism of Schur measures presented in [Ok1] is complemented by explicit computations in [BOk, §4]. One more derivation of the kernel $\underline{K}_{z,z',\xi}(x, y)$ was recently suggested in [BOS].

3. In general, kernels of the form

$$\frac{P(x)Q(y) - Q(x)P(y)}{x - y}$$

are called *integrable kernels*, in accordance to the terminology of [IIS], [De1], [B]. In our case P and Q are expressed through the Gauss hypergeometric function, this is why we called $\underline{K}_{z,z',\xi}(x, y)$ the *discrete hypergeometric kernel*.

²It is worth noting that this claim is no longer true for negative $x + a$, because then the hypergeometric function in the numerator of (2.1) has a singularity compensated by the gamma function in the denominator, and the order at $\xi = 0$ has to be evaluated in a more sophisticated way.

4. The derivation of (3.12) from (3.3) is quite similar to the standard derivation of the Christoffel–Darboux formula for an arbitrary system of orthogonal polynomials. Since, as explained in §2, the functions ψ_a are closely related to the Meixner polynomials, this similarity is not surprising.

5. Once we know that the functions ψ_a form an orthonormal basis (Proposition 2.4), the series expression (3.3) for the kernel $\underline{K}_{z,z',\xi}(x,y)$ immediately implies that it is a projection kernel. This fact was first proved in [BO5, §5] in a different way.

6. The series representation (3.3) is equivalent to formula (3.16) in [Ok2]. A double contour representation of various correlation kernels related to Schur measures appeared earlier in [Bok].

7. Thus, almost all the results obtained in this section were already known. What is really new in our paper is the approach to their derivation based on the relationship to the Meixner polynomials. In [BO7] we apply the same approach to a more complex (dynamical) model.

8. One more novelty of the present work is appearance of the difference operator D ; its importance becomes especially clear in the study of the dynamical model, see [BO7].

4. REMARKS ON A RELATIONSHIP TO KRAWTCHOUK POLYNOMIALS

There exists one more possible choice of basic parameters z , z' , and ξ leading to a family of probability measures on \mathbb{Y} : Namely, parameters z, z' should be nonzero integers of opposite sign, while ξ should be a negative real number (thus, instead of assuming $\xi \in (0, 1)$ we now require $\xi < 0$).

Indeed, let $z = N$ and $z' = -N'$, where N and N' are positive integers, and let $\xi < 0$. Then the weight $M_{z,z',\xi}(\lambda)$, as defined in (1.3), vanishes unless $\ell(\lambda) \leq N$ and $\ell(\lambda') \leq N'$, that is, λ must be contained in the rectangle $N \times N'$. For such diagrams λ , we have $(z)_\lambda > 0$ while the sign of $(z')_\lambda$ equals $(-1)^{|\lambda|}$ (see (1.2)). Since, the sign of $\xi^{|\lambda|}$ also equals $(-1)^{|\lambda|}$, we have $(z')_\lambda \xi^{|\lambda|} > 0$. Therefore, $M_{z,z',\xi}(\lambda) > 0$. The sum of all the weights is still equal to 1, so that we obtain an additional family of probability measures on \mathbb{Y} . Let us call it the *second degenerate series* of z -measures. Its existence was pointed out in [BO5, Example 1.6].

Let L be a positive integer and $p \in (0, 1)$. The *Krawtchouk weight function* with parameters (p, L) is defined on the finite set $\{0, 1, \dots, L\}$ by

$$W_{p,L}^{\text{Krawtchouk}}(\tilde{x}) = \binom{L}{\tilde{x}} p^{\tilde{x}} (1-p)^{L-\tilde{x}}, \quad \tilde{x} = 0, \dots, L.$$

The orthogonal polynomials with this weight are called the *Krawtchouk polynomials*, see [KS, §1.10]. Let us denote them as $\mathfrak{K}_n(\tilde{x}; p, L)$, where n is the degree of the polynomial.

The next claim is a counterpart of Proposition 1.4 and can be checked directly:

Proposition 4.1. *Under correspondence (1.5), the z -measure of the second degenerate series with parameters $(z = N, z' = -N', \xi)$, where $N, N' \in \{1, 2, \dots\}$ and $\xi < 0$, turns into the N -point Krawtchouk orthogonal polynomial ensemble with parameters*

$$p = \frac{\xi}{\xi - 1}, \quad L = N + N' - 1.$$

Note that our assumption $\xi < 0$ implies $0 < p < 1$.

The Krawtchouk polynomials $\mathfrak{K}_n(\tilde{x}; p, L)$ are close relatives of the Meixner polynomials $\mathfrak{M}_n(\tilde{x}; \beta, \xi)$: both families of polynomials can be defined by the same analytic expression involving the Gauss hypergeometric function, only the ranges of the parameters are different. The correspondence between the families can be formally written as follows:

$$\mathfrak{K}_n(\tilde{x}; p, L) = \mathfrak{M}_n(\tilde{x}; -L, \frac{p}{p-1}),$$

see the very end of §1.10 in [KS].

Claim 4.2. *In all arguments of the present paper that rely on the Meixner polynomials and the Meixner ensembles one could equally well use the Krawtchouk polynomials and the corresponding ensembles.*

Notice that any z -measure $M_{N,-N',\xi}$ of the second degenerate series can be written as a mixture of certain probability measures $M_{N,-N'}^{(n)}$ living on the finite sets \mathbb{Y}_n , $0 \leq n \leq NN'$, cf. Remark 1.5. Only now the “mixing distribution” on n ’s is not the negative binomial distribution but the ordinary binomial distribution with weights $\binom{NN'}{n} p^n (1-p)^{NN'-n}$, where $p = \frac{\xi}{\xi-1}$.

The measures $M_{N,-N'}^{(n)}$ admit a nice interpretation: Let $\square_{N,N'}$ denote the rectangular Young diagram with N rows and N' columns. Each standard tableau T of shape $\square_{N,N'}$ can be viewed as a sequence of growing Young diagrams

$$T = (\emptyset = \lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(n)}, \dots, \lambda^{(NN')} = \square_{N,N'}),$$

where $\lambda^{(n)} \in \mathbb{Y}_n$.

Proposition 4.3. *Let $\{T\}$ be the set of all standard tableaux of shape $\square_{N,N'}$ equipped with the uniform probability measure. The push-forward of this measure under the projection $T \mapsto \lambda^{(n)}$ coincides with $M_{N,-N'}^{(n)}$.*

In this form, the measures $M_{N,-N'}^{(n)}$ appeared in [PR]. A slightly different (but essentially equivalent) interpretation can be found in [BO9, §5].

Notice that the measures $M_{z,z'}^{(n)}$ mentioned in Remark 1.5 can be obtained from the measures $M_{N,-N'}^{(n)}$ by analytic continuation in the parameters z, z' : this approach is developed in [BO3].

Finally, notice that the material of this section is also related to the model considered in [GTW].

REFERENCES

- [B] A. Borodin, *Riemann–Hilbert problem and the discrete Bessel kernel*, Intern. Math. Research Notices (2000), no. 9, 467–494; [arXiv: math.CO/9912093](#).
- [BOk] A. Borodin and A. Okounkov, *A Fredholm determinant formula for Toeplitz determinants*, Integral Equations Oper. Theory **37** (2000), 386–396; [arXiv: math.CA/9907165](#).
- [BOO] A. Borodin, A. Okounkov and G. Olshanski, *Asymptotics of Plancherel measures for symmetric groups*, J. Amer. Math. Soc. **13** (2000), 481–515; [arXiv: math.CO/9905032](#).
- [BO1] A. Borodin and G. Olshanski, *Point processes and the infinite symmetric group*, Math. Research Lett. **5** (1998), 799–816; [arXiv: math.RT/9810015](#).
- [BO2] A. Borodin and G. Olshanski, *Distributions on partitions, point processes and the hypergeometric kernel*, Comm. Math. Phys. **211** (2000), 335–358; [arXiv: math.RT/9904010](#).
- [BO3] A. Borodin and G. Olshanski, *Harmonic functions on multiplicative graphs and interpolation polynomials*, Electronic J. Comb. **7** (2000), paper #R28; [math/9912124](#).

- [BO4] A. Borodin and G. Olshanski, *Z-Measures on partitions, Robinson–Schensted–Knuth correspondence, and $\beta = 2$ random matrix ensembles*, In: Random matrix models and their applications (P. Bleher and A. Its, eds). Cambridge University Press. Mathematical Sciences Research Institute Publications **40**, 2001, 71–94; [arXiv:math.CO/9905189](#).
- [BO5] A. Borodin and G. Olshanski, *Random partitions and the Gamma kernel*, Adv. Math. **194** (2005), no. 1, 141–202; [arXiv:math-ph/0305043](#).
- [BO6] A. Borodin and G. Olshanski, *Z-measures on partitions and their scaling limits*, European Journal of Combinatorics **26** (2005), no. 6, 795–834; [arXiv:math-ph/0210048](#).
- [BO7] A. Borodin and G. Olshanski, *Markov processes on partitions*, Prob. Theory and Related Fields **135** (2006), no. 1, 84–152.
- [BO8] A. Borodin and G. Olshanski, *Stochastic dynamics related to Plancherel measure on partitions*, In: Representation Theory, Dynamical Systems, and Asymptotic Combinatorics (V. Kaimanovich and A. Lodkin, eds), Amer. Math. Soc. Translations–Series 2: Advances in the Mathematical Sciences, vol. **217**, 2006; [arXiv:math-ph/0402064](#).
- [BO9] A. Borodin and G. Olshanski, *Asymptotics of Plancherel-type random partitions*, preprint 2006.
- [BOS] A. Borodin, G. Olshanski, and E. Strahov, *Giambelli compatible point processes*, Adv. Appl. Math. **37**, 209–248; [arXiv:math-ph/0505021](#).
- [De1] P. Deift, *Integrable operators*, In: Differential operators and spectral theory: M. Sh. Birman’s 70th anniversary collection (V. Buslaev, M. Solomyak, D. Yafaev, eds.), American Mathematical Society Translations, ser. 2, v. 189, Providence, R.I.: AMS, 1999, pp. 69–84.
- [De2] P. Deift, *Orthogonal polynomials and random matrices: a Riemann–Hilbert approach*, Reprint of the 1998 original, American Mathematical Society, Providence, RI, 2000.
- [Er] A. Erdelyi (ed.), *Higher transcendental functions. Bateman Manuscript Project, vol. I*, McGraw-Hill, New York, 1953.
- [GTW] J. Gravner, C. A. Tracy, and H. Widom, *Limit theorems for height fluctuations in a class of discrete space and time growth models*, J. Statist. Phys. **102** (2001), no. 5–6, 1085–1132.
- [Gr] F. A. Grünbaum, *The bispectral problem: an overview*, In: Special functions 2000: current perspective and future directions (J. Bustoz et al., eds). NATO Sci. Ser. II Math. Phys. Chem., vol. **30**, Kluwer Acad. Publ., Dordrecht, 2001, pp. 129–140.
- [IIS] A. R. Its, A. G. Izergin, V. E. Korepin, N. A. Slavnov, *Differential equations for quantum correlation functions*, Intern. J. Mod. Phys. **B4** (1990), 10037–1037.
- [Jo1] K. Johansson, *Shape fluctuations and random matrices*, Comm. Math. Phys. **209** (2000), 437–476 [arXiv:math.CO/9903134](#).
- [Jo2] K. Johansson, *Discrete orthogonal polynomial ensembles and the Plancherel measure*, Ann. of Math. (2) **153** (2001), no. 1, 259–296; [arXiv:math.CO/9906120](#).
- [Jo3] K. Johansson, *Non-intersecting paths, random tilings and random matrices*, Probab. Theory Related Fields **123** (2002), no. 2, 225–280; [arXiv:math.PR/0011250](#).
- [KOV1] S. Kerov, G. Olshanski, and A. Vershik, *Harmonic analysis on the infinite symmetric group. A deformation of the regular representation*, Comptes Rend. Acad. Sci. Paris, Sér. I **316** (1993), 773–778.
- [KOV2] S. Kerov, G. Olshanski, and A. Vershik, *Harmonic analysis on the infinite symmetric group*, Invent. Math. **158** (2004), 551–642; [arXiv:math.RT/0312270](#).
- [KS] R. Koekoek and R. F. Swarttouw, *The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue*, Delft University of Technology, Faculty of Information Technology and Systems, Department of Technical Mathematics and Informatics, Report no. 98-17, 1998, available via <http://aw.twi.tudelft.nl/~koekoek/askey.html>.
- [Kö] W. König, *Orthogonal polynomial ensembles in probability theory*, Probability Surveys **2** (2005), 385–447.
- [Ma] I. G. Macdonald, *Symmetric functions and Hall polynomials*, 2nd edition, Oxford University Press, 1995.
- [MJD] T. Miwa, M. Jimbo, E. Date, *Solitons: Differential equations, symmetries and infinite dimensional algebras*, Cambridge Univ. Press, Cambridge, 2000.
- [Ni] S. M. Nishigaki, *Level spacings at the metal–insulator transition in the Anderson Hamiltonians and multifractal random matrix ensembles*, Phys. Rev. E **59** (1999), 2853–2862; [arXiv:cond-mat/9809147](#).

- [Ok1] A. Okounkov, *Infinite wedge and measures on partitions*, Selecta Math. **7** (2001), 1–25; [arXiv:math.RT/9907127](#).
- [Ok2] A. Okounkov, *SL(2) and z-measures*, in: Random matrix models and their applications (P. M. Bleher and A. R. Its, eds). Mathematical Sciences Research Institute Publications **40**, Cambridge Univ. Press, 2001, pp. 407–420; [arXiv:math.RT/0002136](#).
- [Ol] G. Olshanski, *An introduction to harmonic analysis on the infinite symmetric group*, In: Asymptotic combinatorics with applications to mathematical physics (A. M. Vershik, ed.), A European mathematical summer school held at the Euler Institute, St. Petersburg, Russia, July 9–20, 2001, Springer Lect. Notes Math. **1815**, 2003, 127–160; [arXiv:math.RT/0311369](#).
- [PR] B. Pittel and D. Romik, *Limit shapes for random square Young tableaux and plane partitions*, [arXiv:math.PR/0405190](#).
- [S] A. Soshnikov, *Determinantal random point fields*, Russian Math. Surveys **55** (2000), no. 5, 923–975 (translation from Uspekhi Mat. Nauk **55** (2000), no. 5 (335), 107–160); [arXiv:math.PR/0002099](#).

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